Week 10 (104)

Wednesday, March 31, 2021 9:02 AM

4.9 Problems

Problem 1. *Use Composite Simpson's rule and the given value of n to approximate the following improper integrals:*

1. $\int_0^1 x^{-1/4}\sin(x)dx$, $n = 4$
2. $\int_0^1 \frac{e^{2x}}{x^{2/5}}dx$, $n = 6$

Problem 2. *Use the transformation $t = x^{-1}$ and the composite Simpson's rule for $n = 4$ to compute:*

$$\int_1^\infty \frac{1}{x^2+9}dx$$

$\int_0^1 \frac{1}{x^{1/2}} < \infty$ $\int_0^1 \frac{1}{x} = \infty$

$\underbrace{\frac{2}{3}x^{3/2}\Big|_0^1}$ $\log|x|\Big|_0^1$

$$\int \frac{f(x)}{x^p} = \int \frac{f(x) - P_4(x) + P_4(x)}{x^p}$$

$$= \underbrace{\int \frac{f(x) - P_4(x)}{x^p}dx}_{G(x)} + \underbrace{\int \frac{P_4(x)}{x^p}dx}_{\text{compute exactly.}}$$

$$\frac{h}{3}\left[G(x_0) + 4G(x_1) + 2G(x_2) + 4G(x_3) + \cdots + 4G(x_{n-1}) + G(x_n)\right]$$

5.1

Problem 3. *Use Theorem 5.4 to show that the following initial-value problems have a unique solution, and find the solution:*

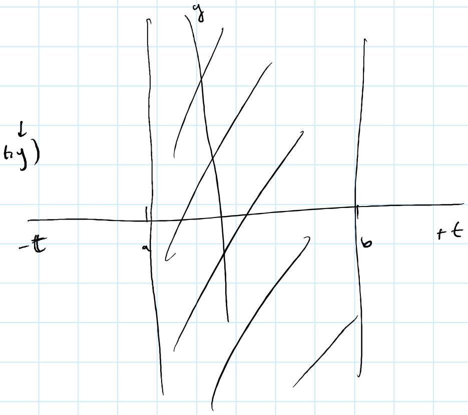
1. $y' = y\cos(t)$, $0 \le t \le 1$, $y(0) = 1$
2. $y' = -\frac{2}{t}y + t^2e^t$, $1 \le t \le 2$, $y(1) = \sqrt{2}e$

Problem 4. *Show that the given equation implicitly defines a solution. Approximate $y(2)$ using Newton's method:*

$$y' = -\frac{y^3+y}{(3y^2+1)t}$$

for $1 \le t \le 2$, $y(1) = 1$. For the equation: $y^3y + yt = 2$

$y' = f(t,y)$



Theorem 5.4 Suppose that $D = \{(t, y) \mid a \le t \le b \text{ and } -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the initial-value problem

$$y'(t) = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

has a unique solution $y(t)$ for $a \le t \le b$. ■

$f(t,y)$ is Lipschitz in D in variable y if $\exists$ constant C s.t.

$$|f(t,y_1) - f(t,y_2)| \le |y_1 - y_2|\, C$$

if $f(t,y)$ has cts derivative in y that is bounded, then $f(t,y)$ is Lipschitz.

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1) $P_4(x)$ for $\sin(x) := x - \frac{x^3}{3!} = x - \frac{x^3}{6}$

$$\int_0^1 \frac{\sin(x)}{x^{1/4}}dx = \underbrace{\int_0^1 \frac{\sin(x) - P_4(x)}{x^{1/4}}dx}_{A} + \underbrace{\int_0^1 \frac{P_4(x)}{x^{1/4}}dx}_{B}$$

$\frac{x}{x^{1/4}} = x^{3/4}$

$\frac{x^3}{x^{1/4}} = \frac{x^{12/4}}{x^{1/4}} = x^{11/4}$

$$B := \int_0^1 \frac{x - \frac{x^3}{6}}{x^{1/4}} = \frac{166}{315}$$

$$A := G(x) = \begin{cases} \frac{\sin(x) - P_4(x)}{x^{1/4}} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$$

integrate by Simpsons

$$\frac{.25}{3}\left(G(0) + 4G(.25) + 2G(.5) + 4G(.75) + G(1)\right)$$

$\cong .00132$

$A + B \cong .528306$

$\cong .00132$

$A + B \cong .528306$

2) $e^{2x} \Rightarrow P_4(x) = 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{3!} + \frac{16x^4}{4!}$

$$\int \frac{e^{2x}}{x^{2/5}}\,dx = \underbrace{\int_0^1 \frac{P_4(x)}{x^{2/5}}}_{\cong\, 4.2012} + \underbrace{\int_0^1 \frac{e^{2x} - P_4(x)}{x^{2/5}}}_{} \leftarrow G(x)$$

$$\frac{(1/6)}{3}\left[G(0) + 4G(1/6) + 2G(2/6) + \cdots + 4G(5/6) + G(1)\right] = .0654$$

$\cong 4.26665$

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2) let $t = \frac{1}{x}$ $\quad x = \frac{1}{t}$ $\quad \frac{dx}{dt} = -\frac{1}{t^2}$ $\quad dx = -\frac{1}{t^2}\,dt$

$$= \int_{x=1}^{x=+\infty} \frac{1}{t^{-2}+9}\,\frac{-1}{t^2}\,dt$$

$$= \int_1^0 \frac{-1}{1+9t^2}\,dt = \int_0^1 \underbrace{\frac{1}{1+9t^2}}\,dt = \frac{1}{3}\arctan(3t)\Big|_0^1$$

$\approx .4112$

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Problem 4. *Show that the given equation implicitly defines a solution. Approximate $y(2)$ using Newton's method:*

$$y' = -\frac{y^3+y}{(3y^2+1)t}$$

for $1 \le t \le 2$, $y(1) = 1$. For the equation: $y^3y + yt = 2$

3) $y' = f(t,y)$ where $f(t,y) = y\cos(t)$

Thm 5.4 $\exists$ unique solution if $f(t,y)$ is Lipschitz in y variable on our region.

$$\left|\frac{f(t,y_1) - f(t,y_2)}{y_1 - y_2}\right| = |\partial_y f(t,\xi)|$$

$y_1 \neq y_2$

Lipschitz constant.

$$|f(t,y_1) - f(t,y_2)| \le |y_1 - y_2|\,|\partial_y f(t,\xi)| \le |y_1 - y_2|\left(\overbrace{\mathrm{Max}\,|\partial_y f(t,y)|}\right)$$

$\partial_y f(t,y) = \partial_y\left[y\cos(t)\right] = \cos(t)$

$\partial_y f(t,y) = \partial_y [y\cos(t)] = \cos(t)$

$|\partial_y f(t,y)| = |\cos(t)| \le 1 \Rightarrow f$ Lip. $\Rightarrow \exists!$ solution.

$y' = y\cos(t)$ $\quad y(0)=1$

$\frac{dy}{dt} = y\cos(t)$

$\frac{dy}{y} = \cos(t)dt$

$\log|y| = \sin(t) + c$

$y = y_0 e^{\sin(t)} = \boxed{e^{\sin(t)}}$

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Integrating factor: $y' + g(t)y = f(t)$

multiply both sides by $e^{\int g}$

3.2) $f(t,y) = \frac{-2}{t}y + t^2e^t$

$|\partial_y f| = \left|\frac{-2}{t}\right| = \left|\frac{2}{t}\right| \le 2$

$1 \le t \le 2$

$1 \ge \frac{1}{t} \ge \frac{1}{2}$

$2 \ge \frac{2}{t} \ge 1$

$y' = \frac{-2}{t}y + t^2e^t$

$y' + \frac{2}{t}y = t^2e^t$

multiply by t^2

$\underbrace{y't^2 + 2ty} = t^4e^t$

$\frac{d}{dt}(yt^2) = t^4e^t$

$yt^2 = \int_1^t t^4e^t + C$

$y = \frac{1}{t^2}\left[\underbrace{\int_1^t t^4e^t} + c\right]$